Національний технічний університет
ДНІПРОВСЬКА ПОЛІТЕХНІКА
1899

Lightweight Design in Mechanical Engineering

Problem 6. Lightweight Design of shafts under combined torsional and bending loads

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Problem 6

Calculate lightweight design of a shaft in equilibrium under combined torsional and bending loads **(Fig. 1).**

Constraints **A and B** are frictionless pinned supports.

Ensure minimum safety margin
$\mathbf{n_{min} = 2}$.

Permissible stress is $\boldsymbol{\sigma_{perm}}$.

Consider a material with high strength-to-density ratio.

Perform material cost comparison.

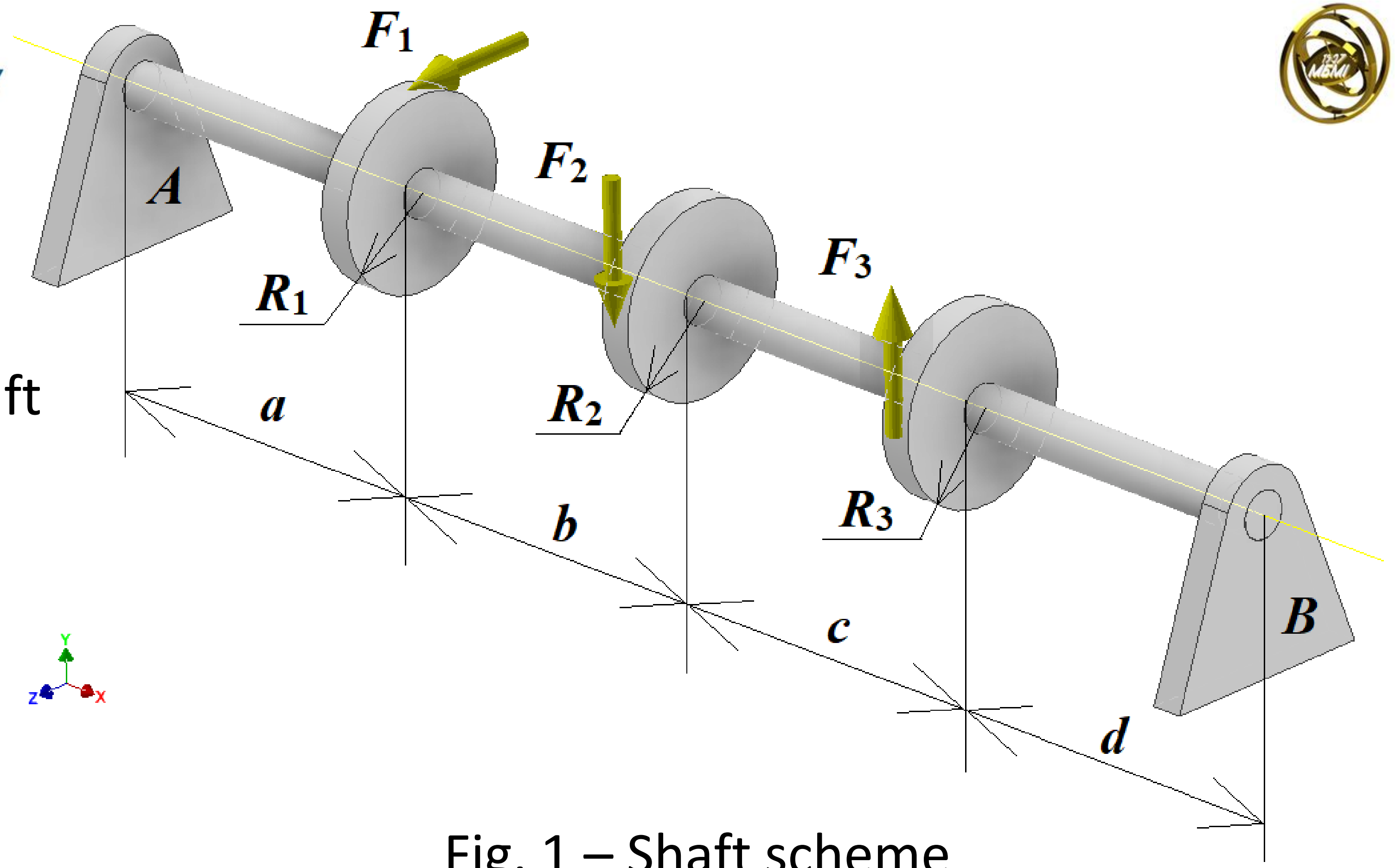

Fig. 1 – Shaft scheme

Force values are $\mathbf{F_1 = 1.5\ kN}$, $\mathbf{F_2 = 0.5\ kN}$, $\mathbf{F_3 = 3\ kN}$.

Dimensions are $\mathbf{R_1 = 0.15\ m}$, $\mathbf{R_2 = 0.15\ m}$, $\mathbf{R_3 = 0.1\ m}$, **a = 0.2 m, b = 0.2 m, c = 0.4 m, d = 0.2 m.**

Problem 6

1. Apply Poinsot's theorem by replacing each tangential force by an axial force and a couple of forces (torque) **(Fig. 2)**.

2. Determine the torques

$T_1 = F_1 \cdot R_1 = 0.225$ kN·m;

$T_2 = F_2 \cdot R_2 = 0.075$ kN·m;

$T_3 = F_3 \cdot R_3 = 0.3$ kN·m.

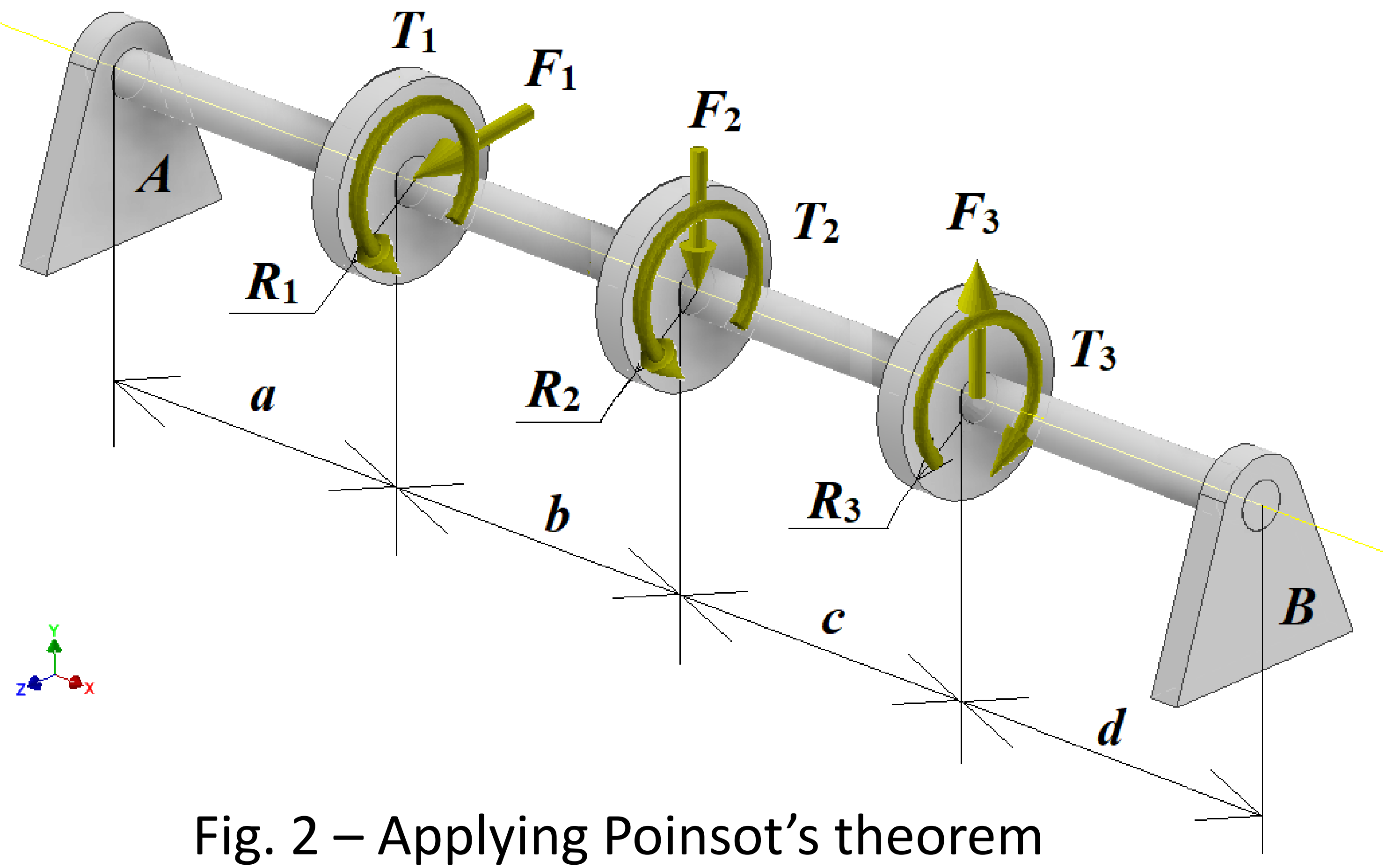

Fig. 2 – Applying Poinsot's theorem

Problem 6

3. Determine internal torques in all shaft sections. Torques in sections with **lengths a and d are 0**. Create free-body diagrams (**Figs. 3, 4**) for sections 1-1 and 2-2. Analyze their equilibrium.

4. Determine the torques

$\Sigma M_x(F_i) = 0$: $-T_1 + T_{1\text{-}1} = 0$;
$T_{1\text{-}1} = 0.225$ kN·m;
$\Sigma M_x(F_i) = 0$: $-T_1 - T_2 + T_{2\text{-}2} = 0$;
$T_{2\text{-}2} = 0.3$ kN·m.

Construct a torque diagram (**Fig. 5**).

$T_{max} = 0.3$ kN·m.

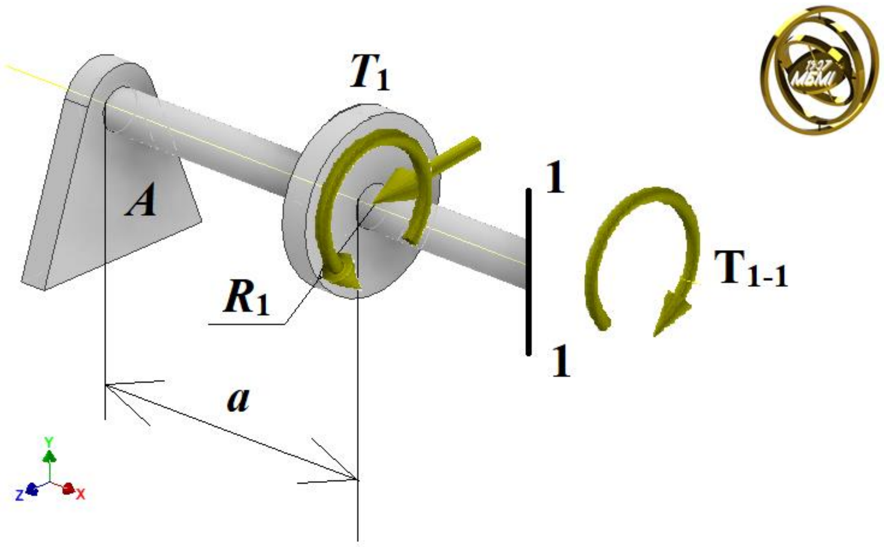

Fig. 3 – Section 1-1

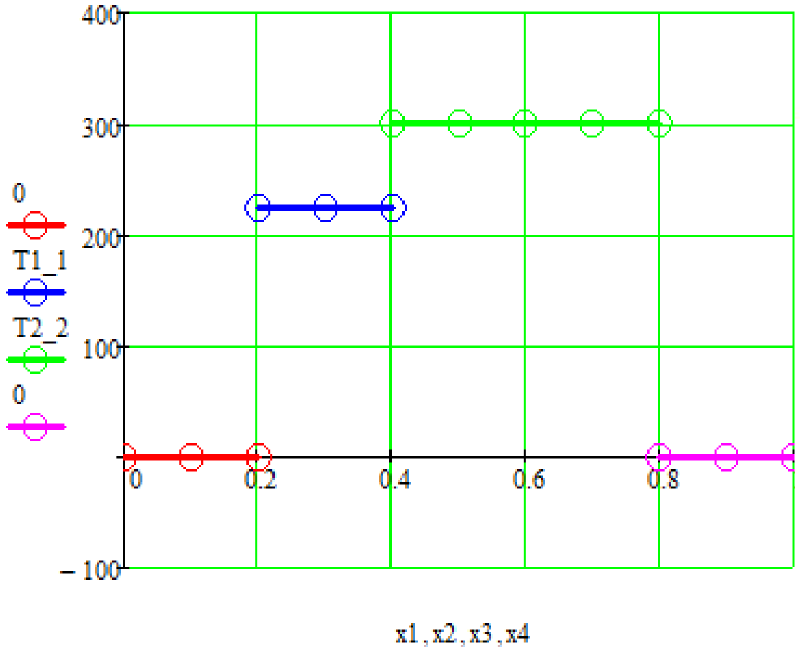

Fig. 5 – Torque diagram

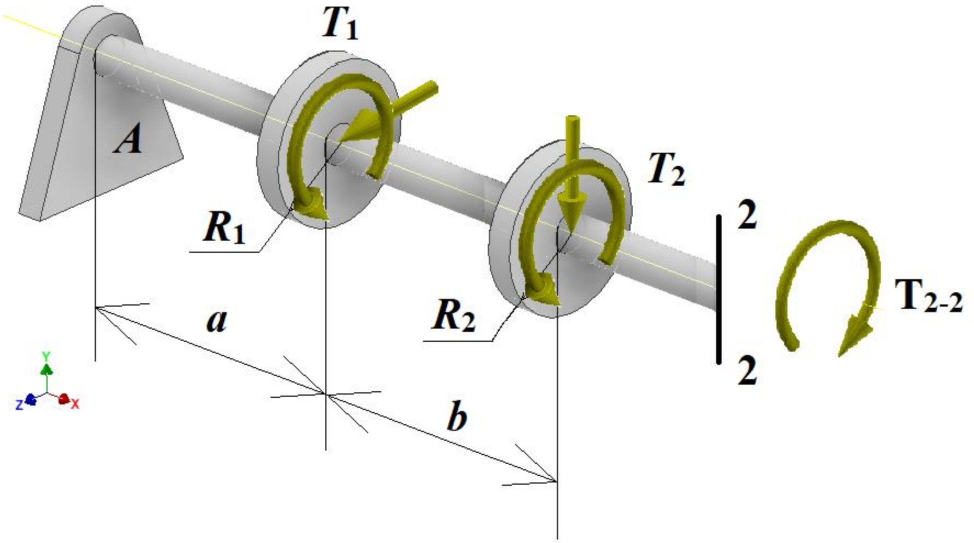

Fig. 4 – Section 2-2

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5. Consider the equilibrium in the **XY plane** using the scheme in **Fig. 6**.

6. Determine reactions of constraints $\mathbf{Y_A}$ and $\mathbf{Y_B}$

$\mathbf{\Sigma M_A(F_i) = 0}$: $-F_2(a + b) + F_3(a + b + c) -$
$-Y_B(a + b + c + d) = 0$;

$\mathbf{Y_B = 2.2}$ **kN;**

$\mathbf{\Sigma M_B(F_i) = 0}$: $Y_A(a + b + c + d) + F_2(c + d) - F_3 d = 0$;

$\mathbf{Y_A = 0.3}$ **kN;**

Check the results

$\mathbf{\Sigma F_{iy} = 0}$: $-Y_A + F_3 - Y_B - F_2 = 0$.

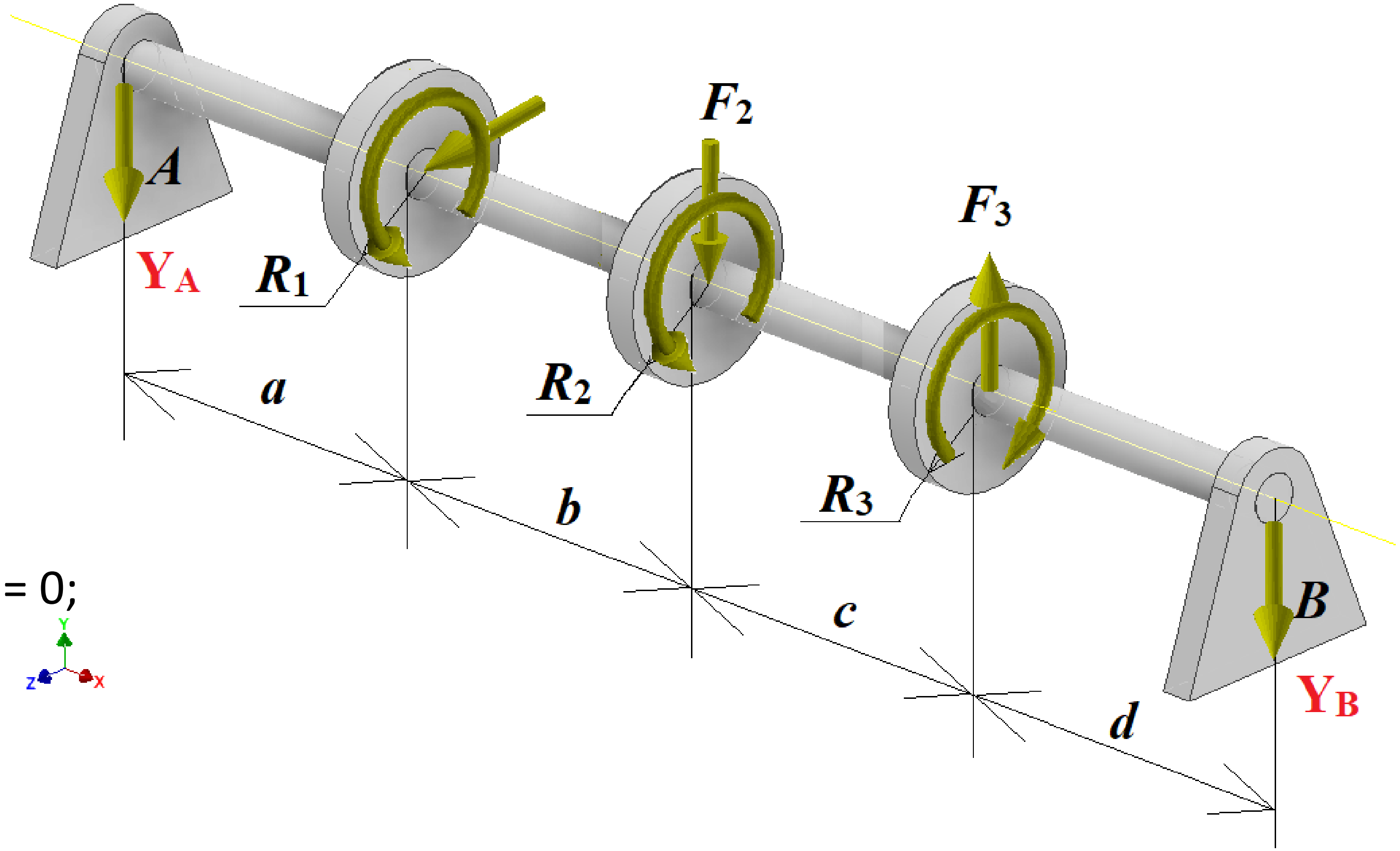

Fig. 6 – Scheme for forces in the **XY plane**

Problem 6

7. Calculate bending moments in the **XY plane** using the schemes in **Fig. 7**.

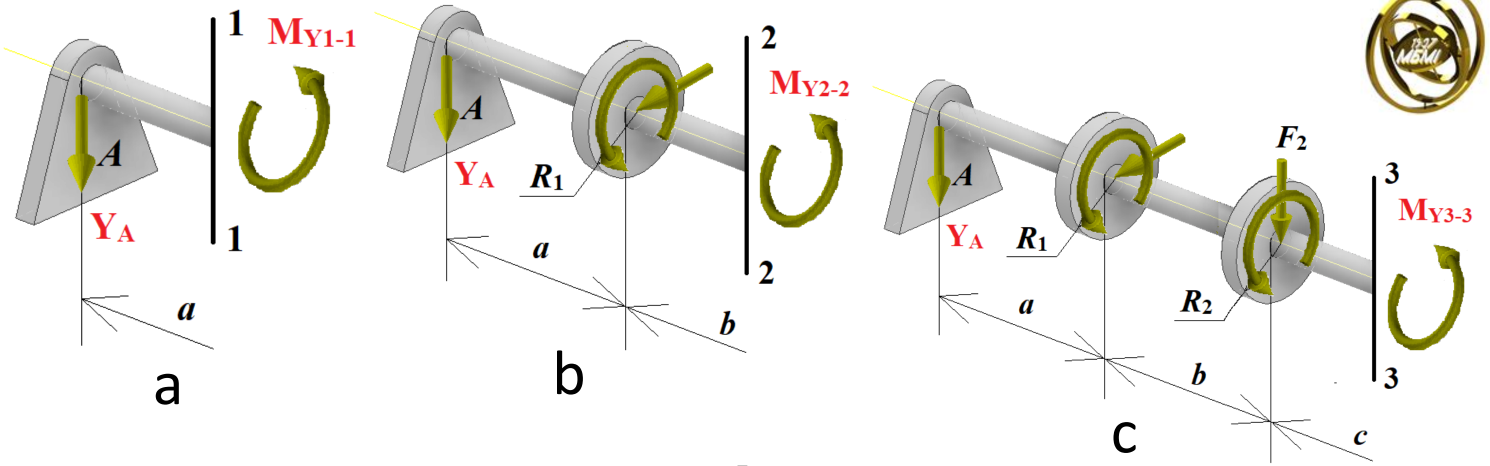
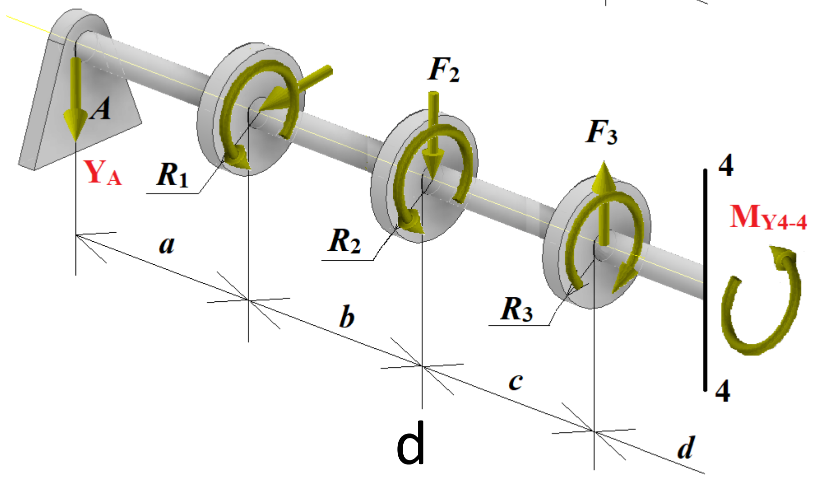

Fig. 7 – Calculation schemes for bending moments in **XY plane** (a – 1-1; b – 2-2; c – 3-3; d – 4-4)

8. Determine bending moments in cross-sections

1-1: $M_{Y1-1} = -Y_A \cdot x_1$;

2-2: $M_{Y2-2} = -Y_A \cdot x_2$;

3-3: $M_{Y3-3} = -Y_A \cdot x_3 - F_2(x_3 - a - b)$;

4-4: $M_{Y4-4} = -Y_A \cdot x_4 - F_2(x_4 - a - b) - F_3(x_4 - a - b - c)$.

Construct a diagram (**Fig. 8**)

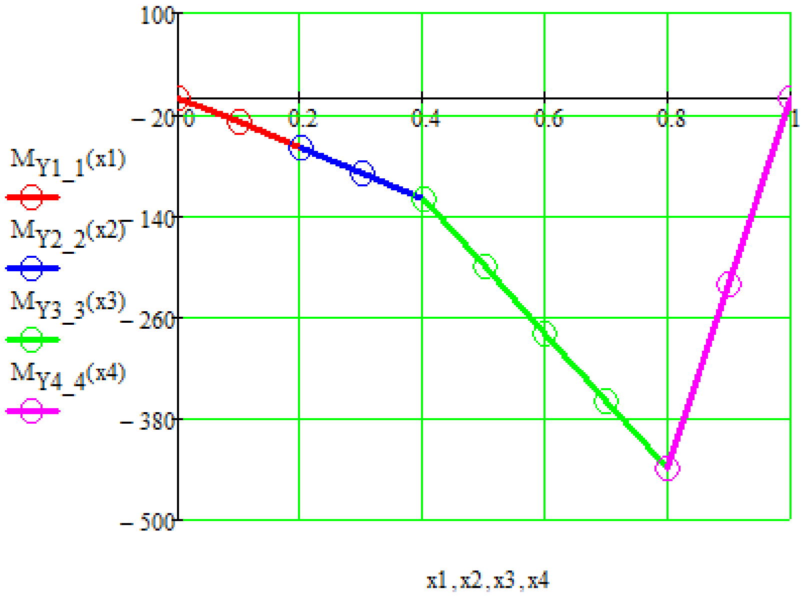

Fig. 8 – Bending moments in **XY plane**

Problem 6

9. Consider the equilibrium in the **XZ plane** using the scheme in **Fig. 9**.

10. Determine reactions of constraints $\mathbf{Z_A}$ and $\mathbf{Z_B}$

$\mathbf{\Sigma M_A(F_i) = 0}$: $Z_B(a + b + c + d) - F_1 a = 0$;

$\mathbf{Z_B = 0.3\ kN;}$

$\mathbf{\Sigma M_B(F_i) = 0}$: $-Z_A(a + b + c + d) +$

$+ F_1(b + c + d) = 0$;

$\mathbf{Z_A = 1.2\ kN.}$

Check the results

$\mathbf{\Sigma F_{iz} = 0}$: $Z_A + Z_B - F_1 = 0$.

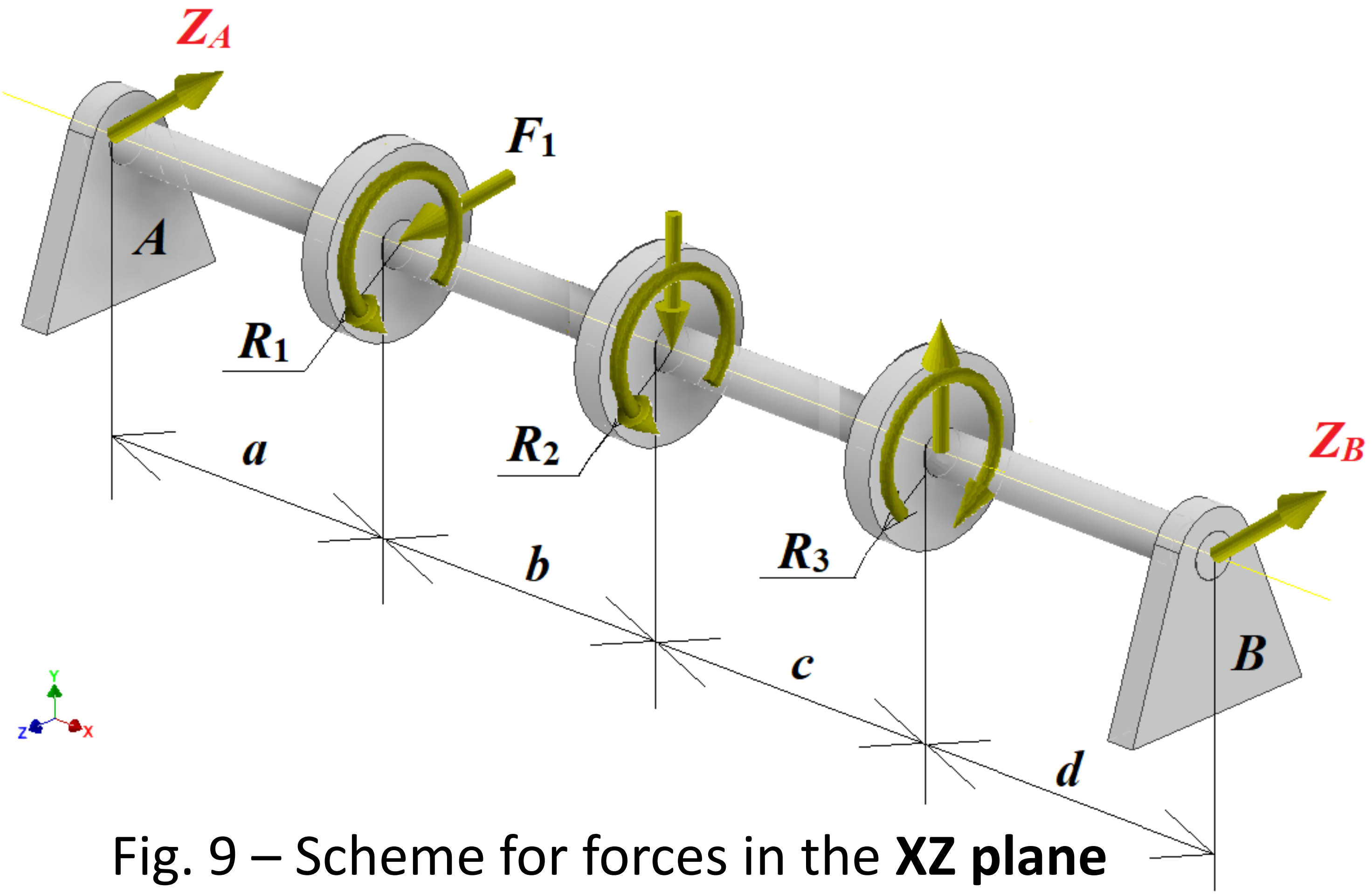

Fig. 9 – Scheme for forces in the **XZ plane**

Problem 6

11. Calculate bending moments in the **XZ plane** using the schemes in **Fig. 10**.

12. Determine bending moments in cross-sections

1-1: $M_{Z1\text{-}1} = Z_A \cdot x_1$;

2-2: $M_{Z2\text{-}2} = Z_A \cdot x_2 - F_1(x_2 - a)$;

3-3: $M_{Z3\text{-}3} = Z_A \cdot x_3 - F_1(x_3 - a)$;

4-4: $M_{Z4\text{-}4} = Z_A \cdot x_4 - F_1(x_4 - a)$.

Construct a diagram (**Fig. 11**)

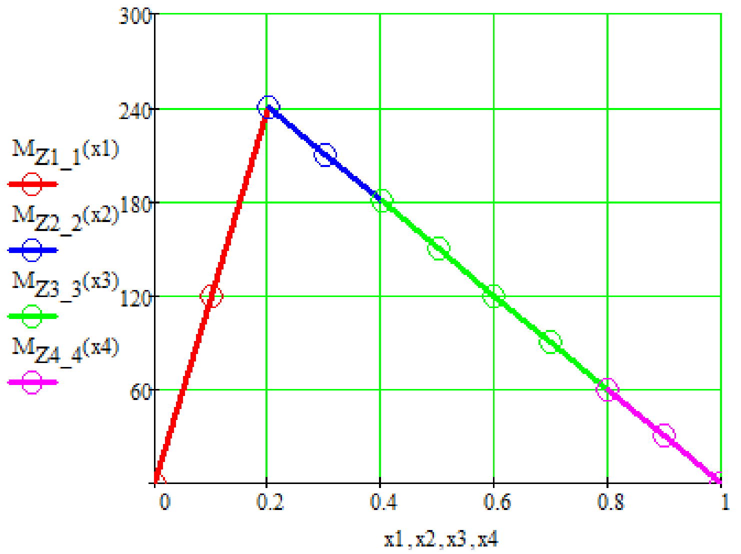

Fig. 11 – Bending moments in **XZ plane**

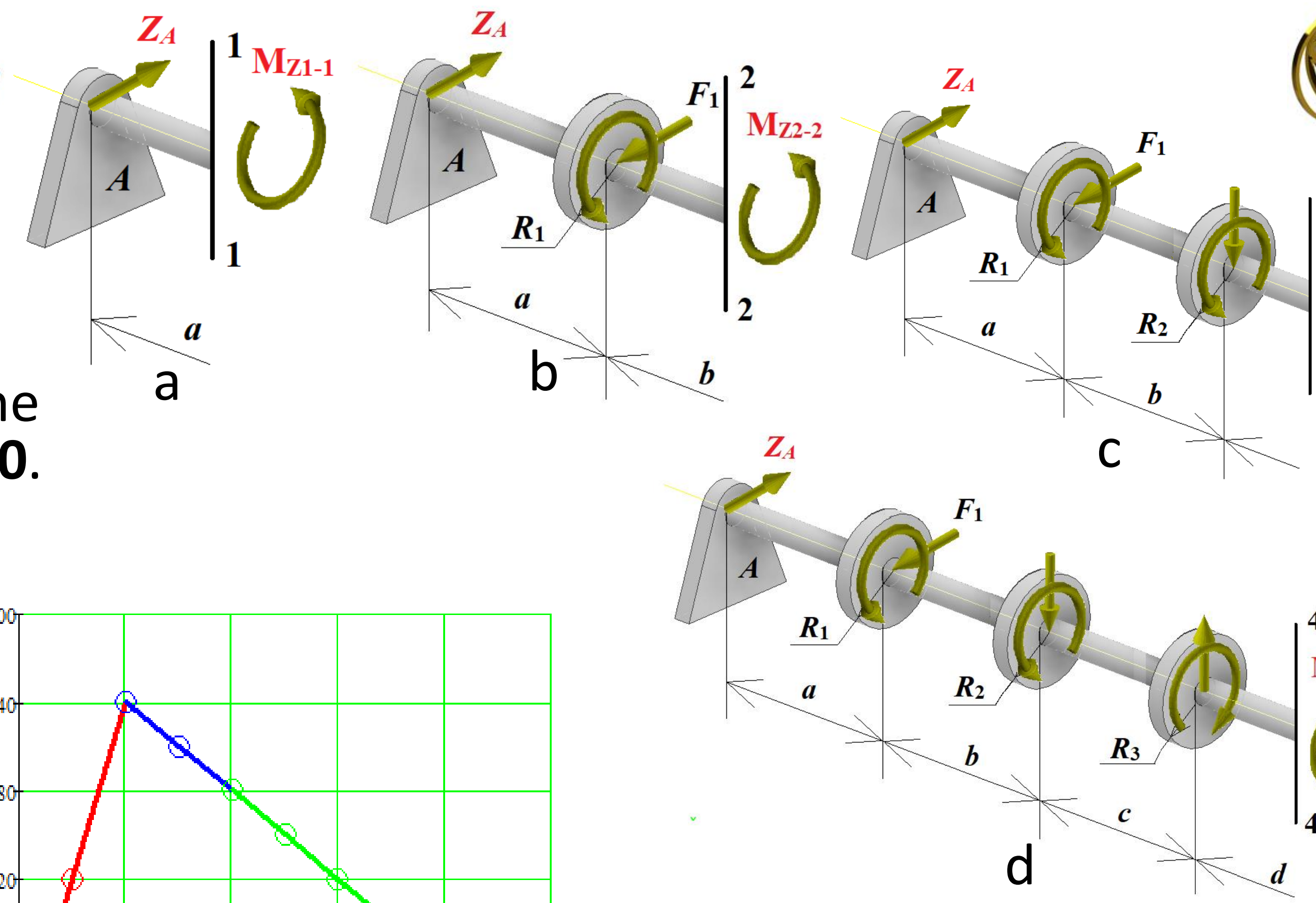

Fig. 10 – Calculation schemes for bending moments in **XZ plane** (a – 1-1; b – 2-2; c – 3-3; d – 4-4)

Problem 6

13. Construct a diagram of resulting bending moments **(Fig. 12)**

$M_{\Sigma 1} = [(M_{Y1\text{-}1})^2 + (M_{Z1\text{-}1})^2]^{0.5}$;
$M_{\Sigma 2} = [(M_{Y2\text{-}2})^2 + (M_{Z2\text{-}2})^2]^{0.5}$;
$M_{\Sigma 3} = [(M_{Y3\text{-}3})^2 + (M_{Z3\text{-}3})^2]^{0.5}$;
$M_{\Sigma 4} = [(M_{Y4\text{-}4})^2 + (M_{Z4\text{-}4})^2]^{0.5}$.

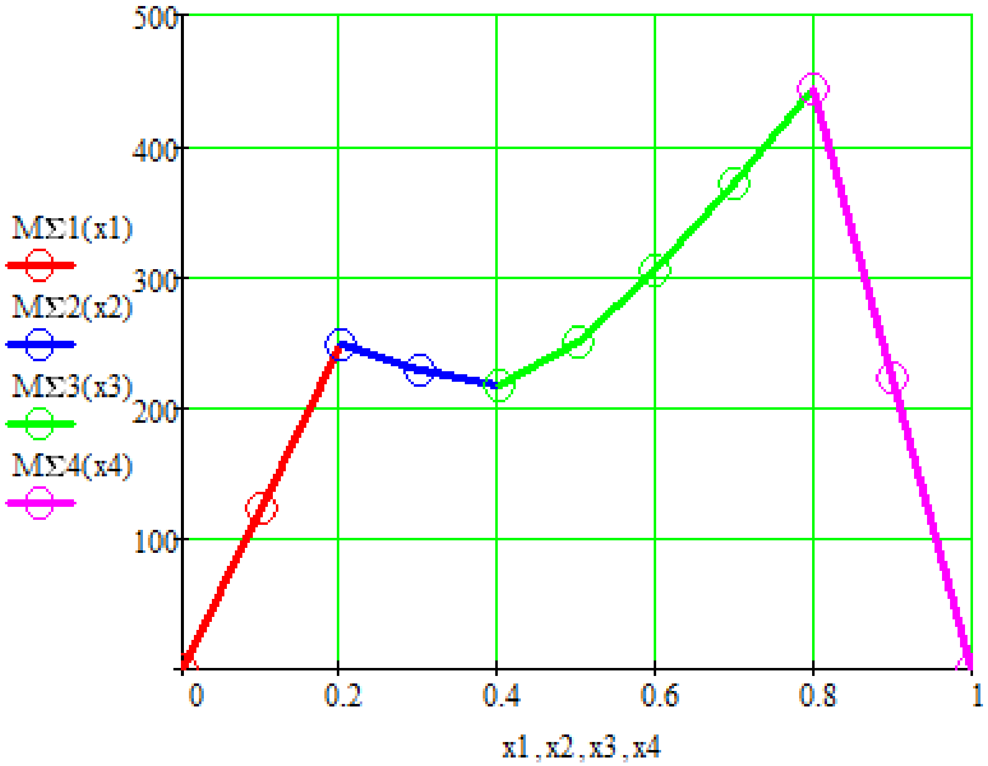

Fig. 12 – Diagram of resulting bending moments

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14. Determine the **reduced moment M_{red}** and solid circular shaft **diameter d_{min}** using maximum distortion energy theory

$$M_{red} = [(M_{\Sigma max})^2 + 0.75(T_{max})^2]^{0.5} = 1.023\ kN{\cdot}m.$$

15. Condition of strength for a shaft is

$$\sigma_{max} = [|M_{red}| / W] \le \sigma_{perm}$$

Section modulus for a solid circular shaft is

$$W = [\pi \cdot (d_{min})^3] / 32$$

16. Minimum diameter for a solid circular shaft

$$d_{min} = [32 \cdot M_{red} / \pi \cdot \sigma_{perm}]^{1/3}$$

Problem 6

17. Consider a **circular solid shaft (round bar)** made of structural steel **S235J2**
Yield strength is $\boldsymbol{\sigma_{yield.St} = 235}$ **MPa;**
Permissible stress $\boldsymbol{\sigma_{perm.St} = \sigma_{yield.St} / n_{min} = 117.5}$ **MPa;**
Density is $\boldsymbol{\rho_{St} = 7850}$ **kg/m³.**

18. Also consider **Titanium Grade 5 (Ti-6Al-4V)**
$\boldsymbol{\sigma_{yield.Ti} = 790}$ **MPa;**
$\boldsymbol{\sigma_{perm.Ti} = \sigma_{yield.Ti} / n_{min} = 395}$ **MPa;**
$\boldsymbol{\rho_{Ti} = 4430}$ **kg/m³.**

19. And consider **Aluminum 7075-T6**
$\boldsymbol{\sigma_{yield.Al} = 500}$ **MPa;**
$\boldsymbol{\sigma_{perm.Al} = \sigma_{yield.Al} / n_{min} = 250}$ **MPa;**
$\boldsymbol{\rho_{Al} = 2810}$ **kg/m³.**

Problem 6

20. Calculate minimum diameters of **circular solid shafts**

$$d_{min.St} = [32 \cdot M_{red} / \pi \cdot \sigma_{perm.St}]^{1/3} = 44.6 \text{ mm};$$

$$d_{min.Ti} = [32 \cdot M_{red} / \pi \cdot \sigma_{perm.Ti}]^{1/3} = 29.8 \text{ mm};$$

$$d_{min.Al} = [32 \cdot M_{red} / \pi \cdot \sigma_{perm.Al}]^{1/3} = 34.7 \text{ mm}.$$

21. Calculate minimum cross-sectional area, which satisfies the safety factor $\mathbf{n_{min} = 2}$

$$A_{min.St} = [\pi \cdot (d_{min.St})^2] / 4 = 1562 \text{ mm}^2;$$

$$A_{min.Ti} = [\pi \cdot (d_{min.Ti})^2] / 4 = 696 \text{ mm}^2;$$

$$A_{min.Al} = [\pi \cdot (d_{min.Al})^2] / 4 = 944 \text{ mm}^2.$$

Problem 6

22. Perform volume calculations

$V_{min.St} = A_{min.St} \cdot (a + b + c + d) = 15.6 \cdot 10^{-4}\ m^3$;
$V_{min.Ti} = A_{min.Ti} \cdot (a + b + c + d) = 7 \cdot 10^{-4}\ m^3$;
$V_{min.Al} = A_{min.Al} \cdot (a + b + c + d) = 9.4 \cdot 10^{-4}\ m^3$.

23. Determine mass of circular shafts

$m_{min.St} = \rho_{St} \cdot V_{min.St} = 12.26$ kg;
$m_{min.Ti} = \rho_{Ti} \cdot V_{min.Ti} = 3.08$ kg;
$m_{min.Al} = \rho_{Al} \cdot V_{min.Al} = 2.65$ kg.

24. Prices per kg of materials are

$p_{St} = 0.728$ €/kg;
$p_{Ti} = 5.97$ €/kg;
$p_{Al} = 2.955$ €/kg.

25. Calculate material cost

$MC_{St} = m_{min.St} \cdot p_{St} = 8.93$ €;
$MC_{Ti} = m_{min.Ti} \cdot p_{Ti} = 18.4$ €;
$MC_{Al} = m_{min.Al} \cdot p_{Al} = 7.84$ €.

Problem 6

Conclusion.

Structure made of **aluminum** is the **lightest** and the **cheapest**, while having the same safety factor as **steel** and **titanium** structures.

Compared to steel, the **aluminum** structure is only **22%** in terms of weight, the **titanium** structure is **25%** of steel structure.

The cost of **aluminum** shaft **is 88%** of the **steel** cost, and **titanium** shaft is **205%** of steel cost.

Therefore, from considerations of just weight and price, aluminum is the rational material.

Thank you for your attention!

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