Lightweight Design in Mechanical Engineering

Problem 4. Lightweight Design of shafts under torsion

Matthias Kröger, Prof. Dr.-Ing. at TUBAF
Serhii Onyshchenko, PhD, assoc. prof. at DUT

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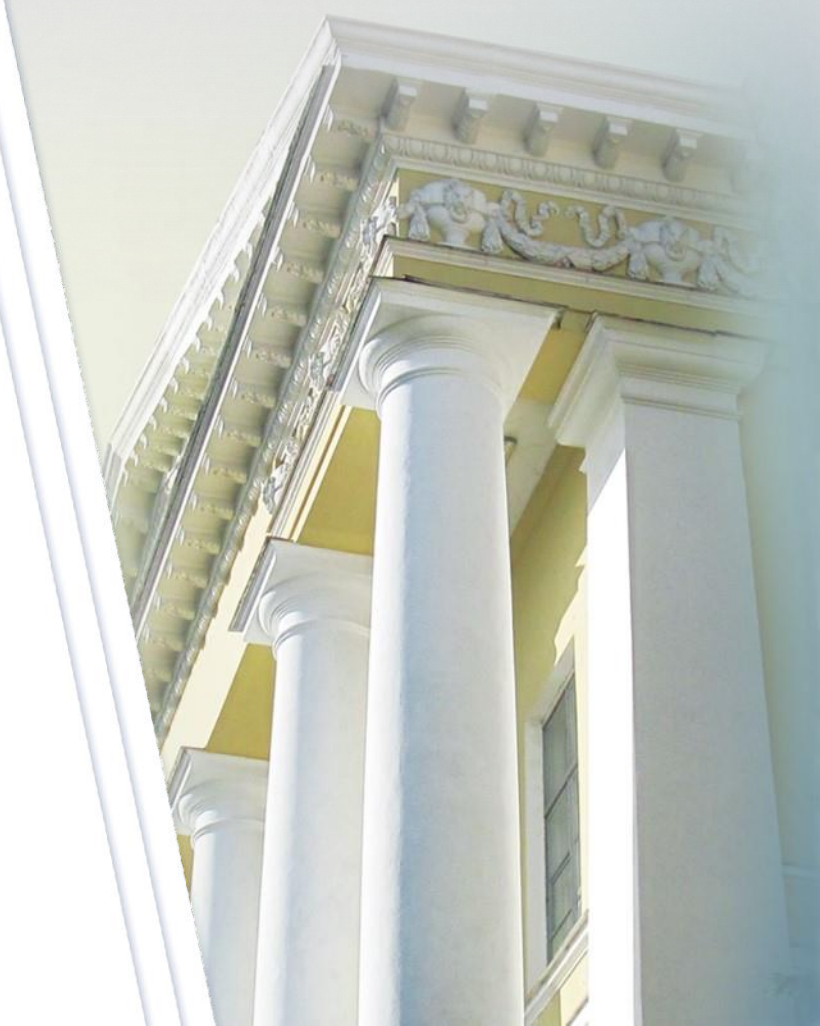

Problem 4

Calculate lightweight design of a shaft **(Fig. 1)** loaded with external torques $\mathbf{T_1}$, $\mathbf{T_2}$, $\mathbf{T_3}$, and $\mathbf{T_4}$. The shaft is in equilibrium and has two pinned constraints A and B, there are no reactions nor friction in constraints. Torque values are $\mathbf{T_1 = 0.5}$ **kN·m**, $\mathbf{T_2 = 1}$ **kN·m**, $\mathbf{T_3 = 2}$ **kN·m** and $\mathbf{T_4 = 1.5}$ **kN·m**.

Dimensions are **a = 1 m, b = 2 m, c = 1 m.**

Ensure minimum safety margin $\mathbf{n_{min} = 2}$.

Permissible stress in members is $\boldsymbol{\tau_{perm}}$.

Consider a material with high strength-to-density ratio.

Perform material cost comparison.

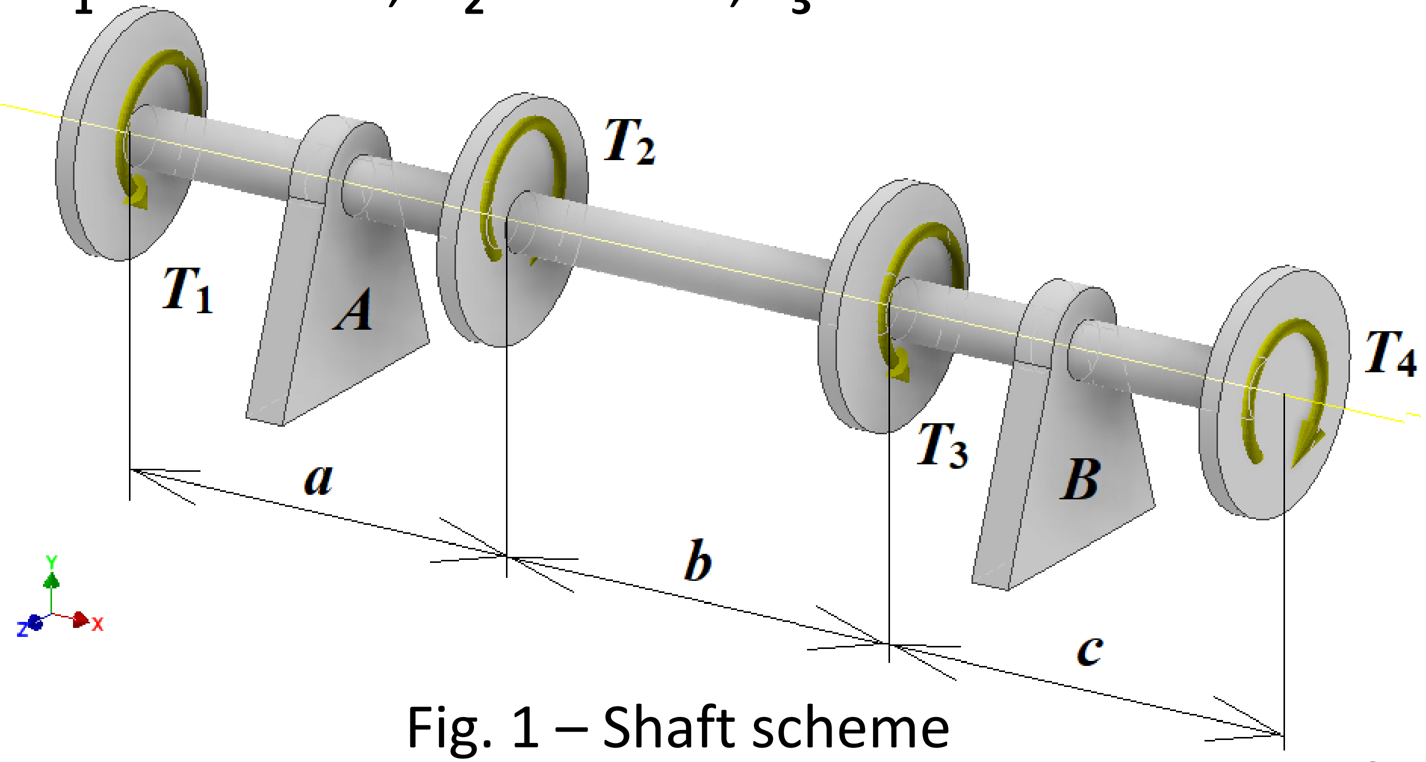

Fig. 1 – Shaft scheme

Problem 4

1. Calculate internal torques in all shaft sections. Torsional resistance in supports A and B is absent, therefore, there must be only 3 different sections. Create free-body diagrams (**Figs. 2-4**) for sections 1-1, 2-2, and 3-3. Analyze their equilibrium.

2. Determine the torques

$\Sigma T_i = 0: -T_1 + T_{1\text{-}1} = 0;$

$T_{1\text{-}1} = 0.5$ kN·m;

$\Sigma T_i = 0: -T_1 + T_2 + T_{2\text{-}2} = 0;$

$T_{2\text{-}2} = -0.5$ kN·m.

$\Sigma T_i = 0: -T_1 + T_2 - T_3 + T_{3\text{-}3} = 0;$

$T_{3\text{-}3} = 1.5$ kN·m.

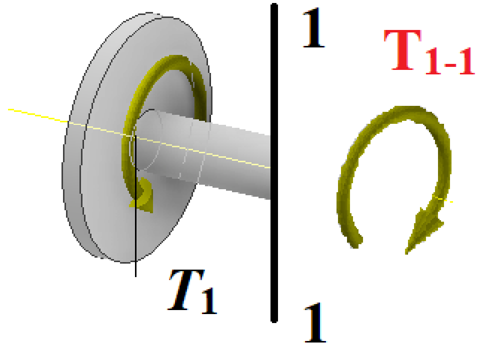

Fig. 2 – Section 1-1

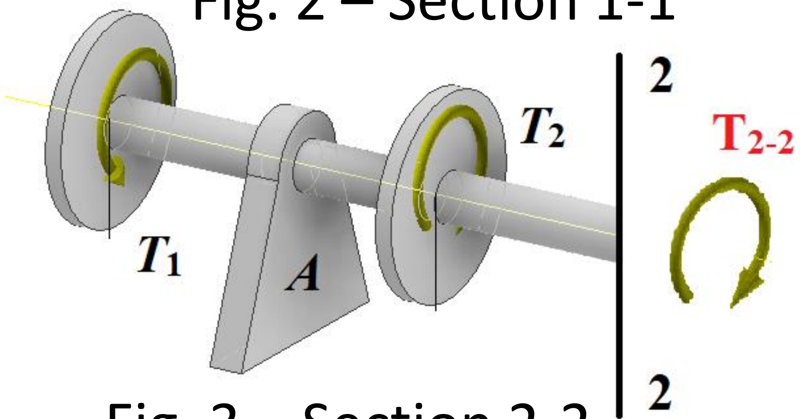

Fig. 3 – Section 2-2

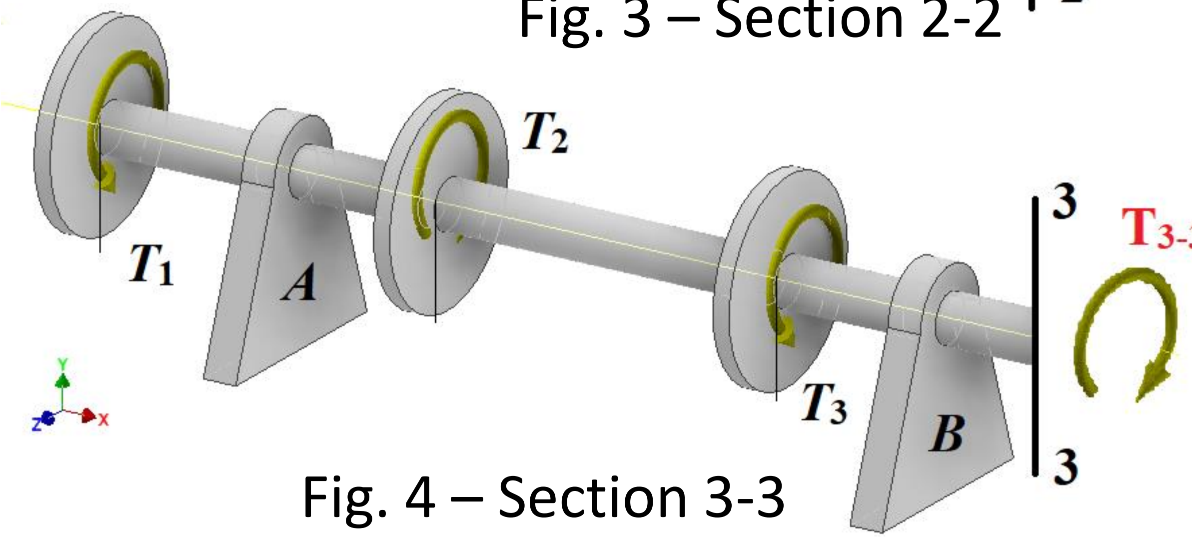

Fig. 4 – Section 3-3

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3. Construct a torque diagram (**Fig. 5**).

T_{max} **= 1.5 kN·m.**

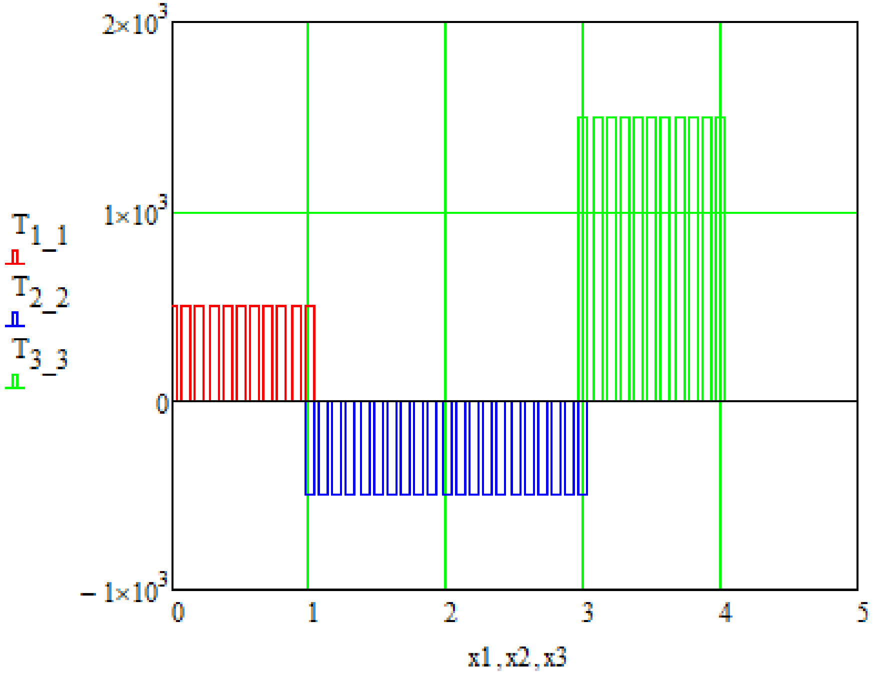

Fig. 5 – Torque diagram

4. Consider the following materials for a shaft

Structural steel S235J2

Yield strength is $\sigma_{yield.St}$ **= 235 MPa;**

Permissible normal stress $\sigma_{perm.St} = \sigma_{yield.St} / n_{min}$ **= 117.5 MPa;**

Permissible shear stress $\tau_{perm.St} = 0.6 \cdot \sigma_{perm.St}$ **= 70.5 MPa;**

Density is ρ_{St} **= 7850 kg/m³;**

Shear modulus **G = 81 GPa.**

Titanium Grade 5 (Ti-6Al-4V)

$\sigma_{yield.Ti}$ **= 790 MPa;**

$\sigma_{perm.Ti} = \sigma_{yield.Ti} / n_{min}$ **= 395 MPa;**

$\tau_{perm.Ti} = 0.6 \cdot \sigma_{perm.Ti}$ **= 237 MPa;**

ρ_{Ti} **= 4430 kg/m³;**

G = 42 GPa.

Aluminum 7075-T6

$\sigma_{yield.Al}$ **= 500 MPa;**

$\sigma_{perm.Al} = \sigma_{yield.Al} / n_{min}$ **= 250 MPa;**

$\tau_{perm.Al} = 0.6 \cdot \sigma_{perm.Al}$ **= 150 MPa;**

ρ_{Al} **= 2810 kg/m³;**

G = 27 GPa.

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5. Formulate the **condition of strength** for a shaft

$\tau_{max} = [\ |T_{max}|\ /\ W_\rho\ ] \leq \tau_{perm.St}$

polar section modulus for a solid circular shaft is

$W_\rho = [\pi \cdot (d_{min})^3]\ /\ 16;$

6. **Calculate minimum diameter** for a solid circular shaft from the **condition of strength**

for steel

$d_{min.str.St} = [16 \cdot T_{max}\ /\ \pi \cdot \tau_{perm.St}]^{1/3} = 47.68$ mm;

for titanium

$d_{min.str.Ti} = [16 \cdot T_{max}\ /\ \pi \cdot \tau_{perm.Ti}]^{1/3} = 31.83$ mm;

for aluminum

$d_{min.str.Al} = [16 \cdot T_{max}\ /\ \pi \cdot \tau_{perm.Al}]^{1/3} = 37.08$ mm.

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7. Formulate the **condition of rigidity** for a shaft

$$|\theta_{max}| = [\ |T_{max}| / G \cdot I_\rho\ ] \leq \theta_{perm}$$

polar second moment of area for a solid circular shaft is

$$I_\rho = [\pi \cdot (d_{min})^4] / 32;$$

permissible angle of torsion per unit length

$$\theta_{perm} = 2 \text{ deg/m} = 0.035 \text{ rad/m}$$

8. **Calculate minimum diameter** for a solid circular shaft from the **condition of rigidity**

for steel

$$d_{min.rig.St} = [32 \cdot T_{max} / G_{St} \cdot \pi \cdot \theta_{perm}]^{1/4} = 48.2 \text{ mm};$$

for titanium

$$d_{min.rig.Ti} = [32 \cdot T_{max} / G_{Ti} \cdot \pi \cdot \theta_{perm}]^{1/4} = 56.8 \text{ mm};$$

for aluminum

$$d_{min.rig.Al} = [32 \cdot T_{max} / G_{Al} \cdot \pi \cdot \theta_{perm}]^{1/4} = 63.4 \text{ mm}.$$

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9. Out of the **two conditions**, the latter **requires larger overall diameters**. Therefore, accept the diameters **from the condition of rigidity** as basic for further calculations.

10. Calculate minimum cross-sectional area, which satisfies the safety factor $\mathbf{n_{min} = 2}$

$$A_{min.St} = [\pi \cdot (d_{min.rig.St})^2] / 4 = 1823\ mm^2;$$

$$A_{min.Ti} = [\pi \cdot (d_{min.rig.Ti})^2] / 4 = 2532\ mm^2;$$

$$A_{min.Al} = [\pi \cdot (d_{min.rig.Al})^2] / 4 = 3158\ mm^2.$$

11. Determine the volumes of shafts

for steel

$$V_{min.St} = A_{min.St} \cdot (a + b + c) = 72.9 \cdot 10^{-4}\ m^3;$$

for titanium

$$V_{min.Ti} = A_{min.Ti} \cdot (a + b + c) = 101.3 \cdot 10^{-4}\ m^3;$$

for aluminum

$$V_{min.Al} = A_{min.Al} \cdot (a + b + c) = 126.3 \cdot 10^{-4}\ m^3.$$

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12. Perform calculations of shaft **(Fig. 6)** mass

$m_{min.St} = \rho_{St} \cdot V_{min.St} = 57.25$ **kg;**
$m_{min.Ti} = \rho_{Ti} \cdot V_{min.Ti} = 44.86$ **kg;**
$m_{min.Al} = \rho_{Al} \cdot V_{min.Al} = 35.49$ **kg.**

13. Prices per kg of materials are

$p_{St} = 0.728$ **€/kg;**
$p_{Ti} = 5.97$ **€/kg;**
$p_{Al} = 2.955$ **€/kg.**

14. Calculate shaft material cost

$MC_{St} = m_{min.St} \cdot p_{St} = 41.68$ **€;**
$MC_{Ti} = m_{min.Ti} \cdot p_{Ti} = 267.86$ **€;**
$MC_{Al} = m_{min.Al} \cdot p_{Al} = 104.89$ **€.**

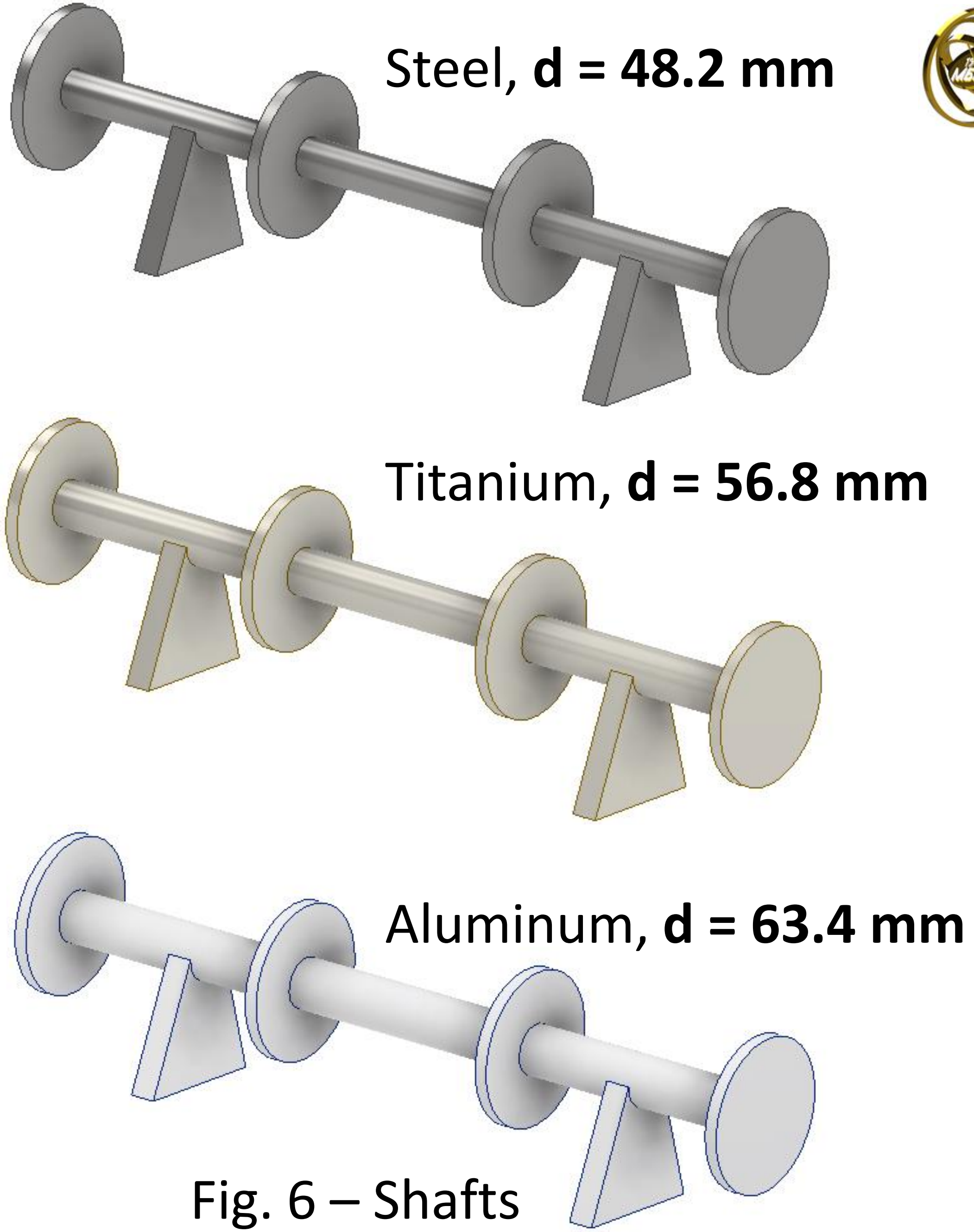

Fig. 6 – Shafts

Problem 4

Conclusion.

Out of the 3 materials – steel, titanium and aluminum – **steel shaft has the minimum diameter**, which satisfies both the condition of strength and rigidity, while ensuring $\mathbf{n_{min} = 2}$. Sufficient **titanium shaft diameter** is **18% larger** than for steel, and **aluminum shaft diameter** is **32% larger** than for steel.

Although, the **mass of the steel shaft** with sufficient diameter **is the largest**, **titanium** shaft is only **78% of steel shaft mass**, and **aluminum** shaft is **62% of steel shaft mass**.

In terms of material cost, **steel shaft is the cheapest**. **Titanium** shaft is **6.4 times more expensive**, and **aluminum** shaft is **2.5 times more costly**.

Therefore, the final decision on material selection must be made based on the **complex evaluation of shaft dimensions, mass and cost**.

Thank you for your attention!

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