Lightweight Design in Mechanical Engineering

Problem 5. Lightweight Design of simply supported and cantilever beams

Matthias Kröger, Prof. Dr.-Ing. at TUBAF
Serhii Onyshchenko, PhD, assoc. prof. at DUT

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Problem 5

Calculate lightweight design of a simply supported beam with a cantilever part **(Fig. 1)** under bending loads.

Ensure minimum safety margin $\mathbf{n_{min} = 2}$.

Permissible stress is $\boldsymbol{\sigma_{perm}}$.

Consider a material with high strength-to-density ratio.

Perform material cost comparison.

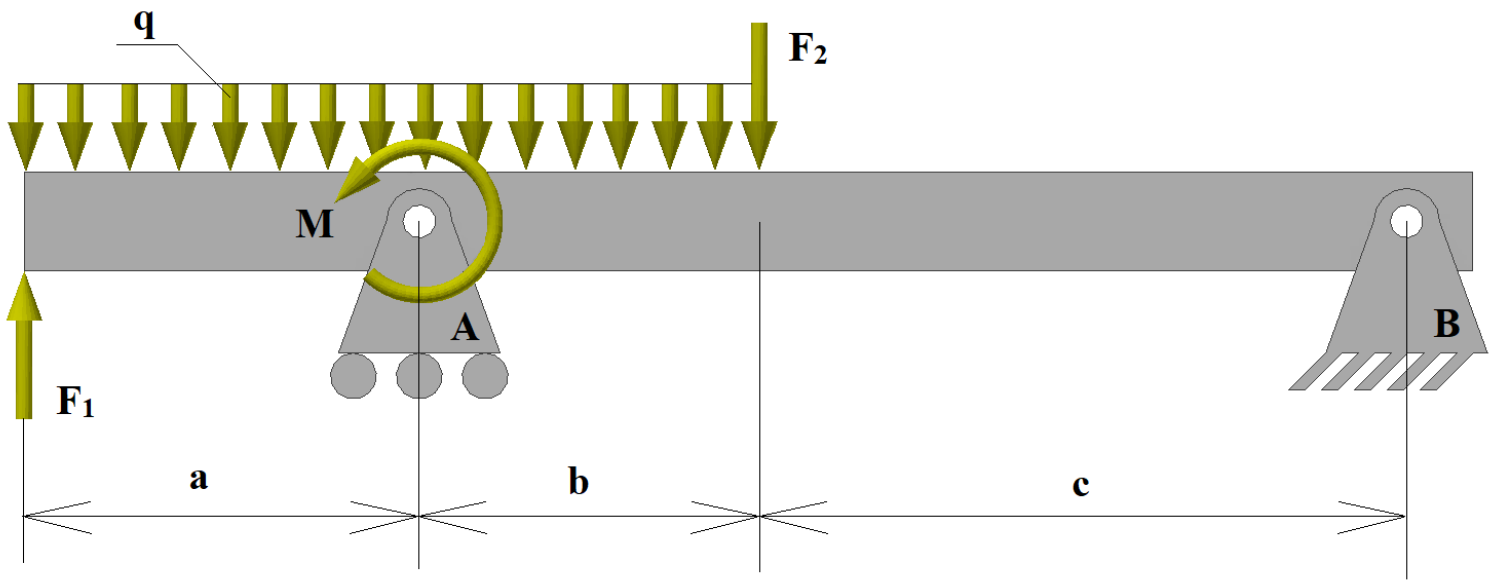

Fig. 1 – Simply supported beam loading scheme

Load values are $\mathbf{F_1 = 4}$ **kN**; $\mathbf{F_2 = 2}$ **kN**; **M = 8 kN·m; q = 4 kN/m.**

Dimensions are **a = 4 m, b = 2 m, c = 2 m.**

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1. Add reactions of constraints R_A and R_B (Fig. 2). Horizontal reactions are absent. Consider beam equilibrium.

2. Construct equations of equilibrium

$\Sigma M_A(F_i) = 0$: $- F_1 \cdot a + q \cdot a \cdot a/2 - - q \cdot b \cdot b/2 - R_B \cdot (b + c) - F_2 \cdot b = 0.$

$\Sigma M_B(F_i) = 0$: $- R_A \cdot (b + c) - - F_1 \cdot (a + b + c) + + q \cdot (a + b) [(a + b)/2 + c] + M + + F_2 \cdot c = 0.$

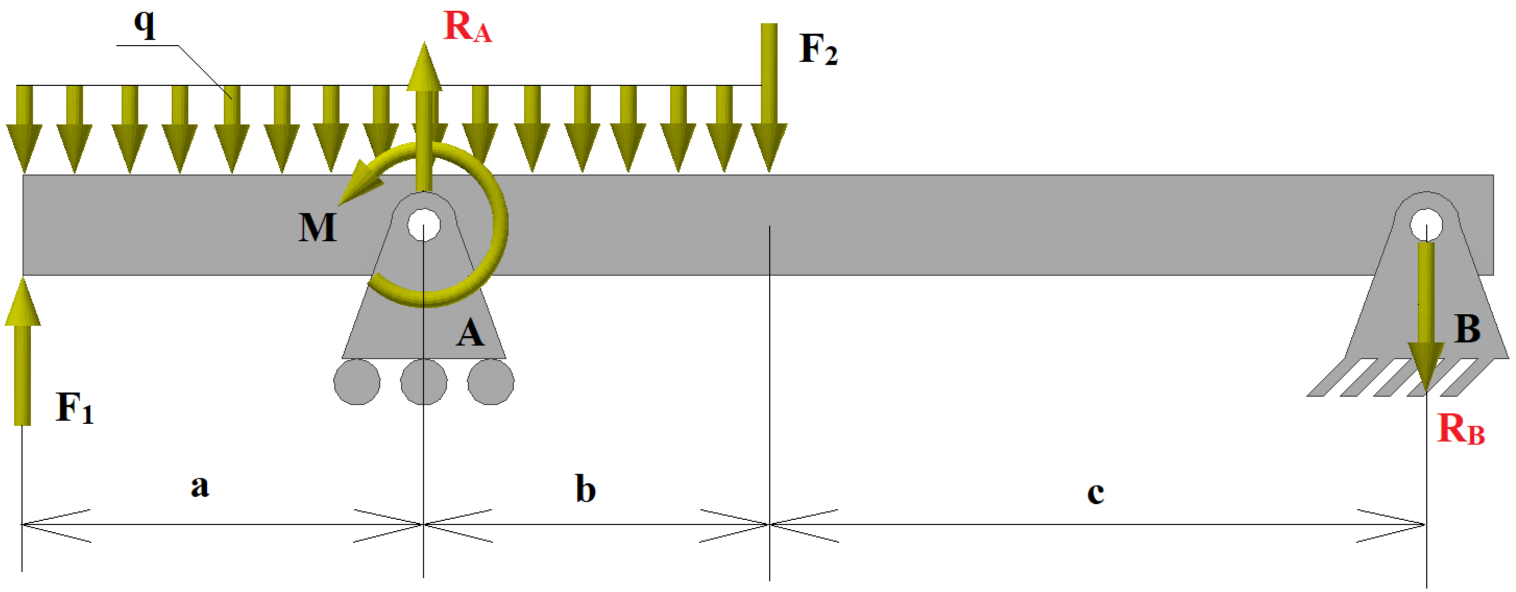

Fig. 2 – Calculation scheme for reactions of constraints

3. Determine the unknown reactions

$R_B = 3$ kN; $R_A = 25$ kN.

4. Check the results

$\Sigma F_{iy} = 0$: $F_1 + R_A - q \cdot (a + b) - R_B - F_2 = 0.$

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5. Construct free-body diagrams of beam sections **(Fig. 3)**.

6. Calculate shear forces and bending moments in beam cross-sections

1-1: $Q_{1-1} = F_1 - q \cdot x_1;$
$M_{1-1} = F_1 \cdot x_1 - q \cdot x_1 \cdot x_1/2;$

2-2: $Q_{2-2} = F_1 - q \cdot x_2 + R_A;$
$M_{2-2} = F_1 \cdot x_2 + R_A(x_2 - a) - q \cdot x_2 \cdot x_2/2 - M;$

3-3: $Q_{3-3} = F_1 - q \cdot (a + b) + R_A - F_2;$
$M_{3-3} = F_1 \cdot x_3 + R_A(x_3 - a) - M -$
$- q \cdot (a + b) \cdot [x_3 - (a + b)/2] -$
$- F_2 \cdot (x_3 - a - b).$

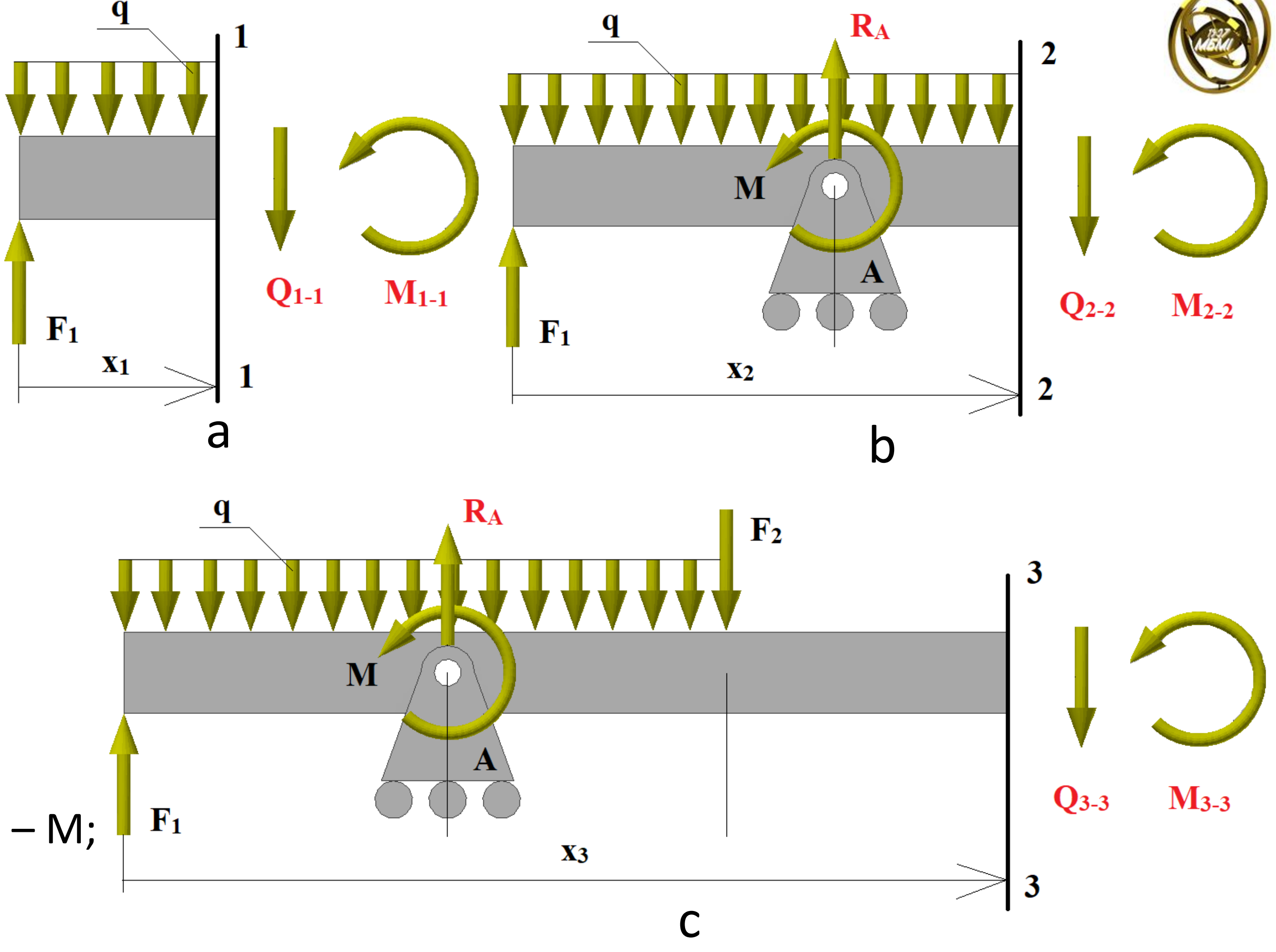

Fig. 3 – Free-body diagrams of beam sections (a – 1-1; b – 2-2; c – 3-3)

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7. Construct diagrams of shear forces **Q** **(Fig. 4)**

and bending moments **M** (**Fig. 5**).

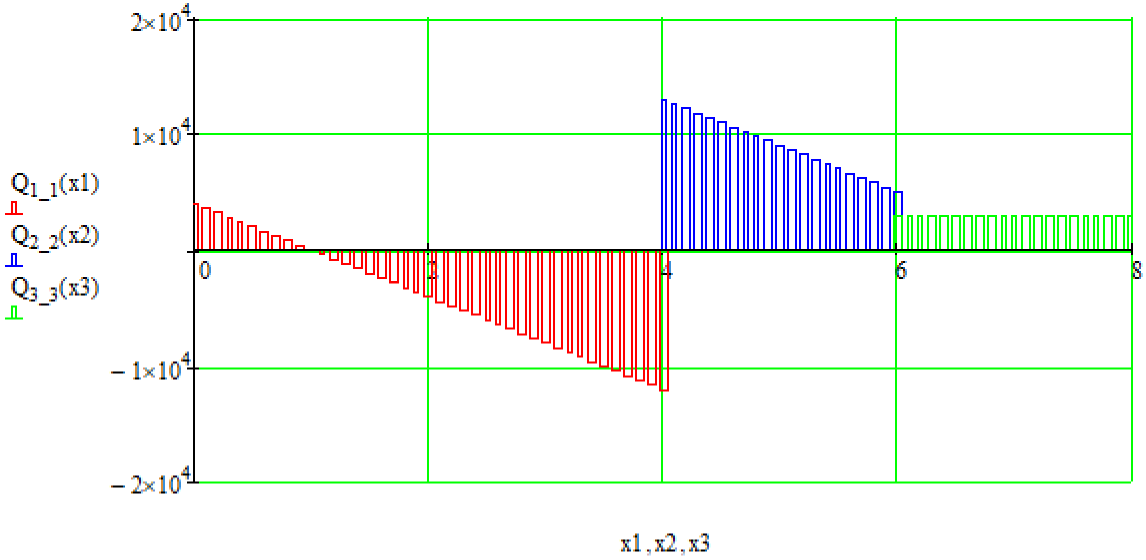

Fig. 4 – Diagram of shear forces

8. Determine maximum values of shear forces $\mathbf{Q_{max}}$ and bending moments $\mathbf{M_{max}}$

$\mathbf{Q_{max} = Q_{2\text{-}2}(a) = 13\ kN;}$

$\mathbf{M_{max} = M_{2\text{-}2}(a) = -24\ kN \cdot m.}$

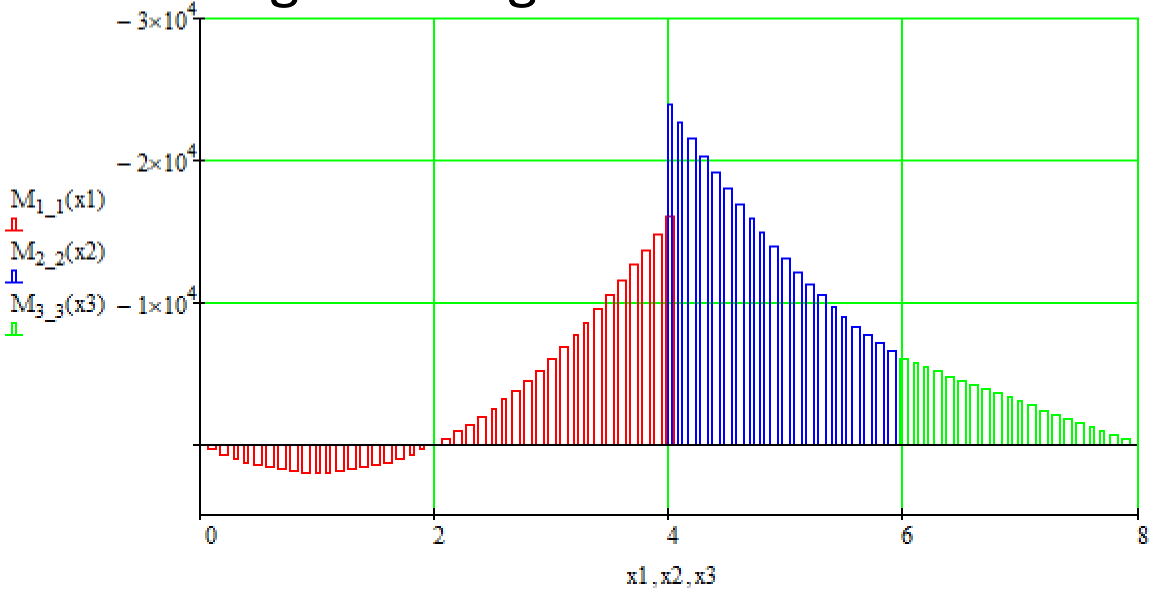

Fig. 5 – Diagram of bending moments

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9. Formulate the **condition of strength** for a beam

$\sigma_{max} = [|M_{max}| / W] \leq \sigma_{perm}$

W – is **section modulus**, it depends on cross-section shape

10. The value of **minimum section modulus** is

$W_{min} = |M_{max}| / \sigma_{perm}$

11. Consider 3 shapes of cross-section – **round**, **rectangular** (h = 2b), and an **I-beam (Fig. 6)**.

Section modulus for a solid round beam is

$W_{round} = [\pi \cdot (d_{min})^3] / 32;$

for rectangular (h = 2b)

$W_{rect} = b \cdot h^2 / 6 = h^3 / 12.$

Section modulus for an **I-beam** is determined from a standard.

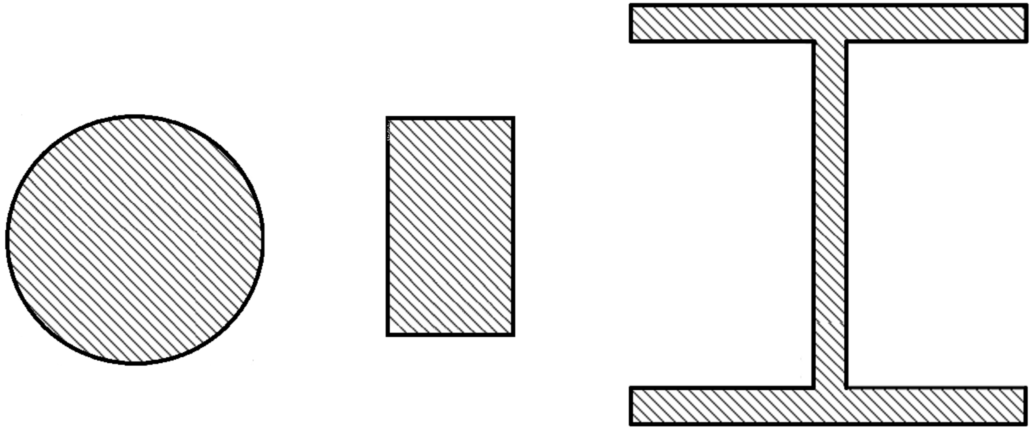

Fig. 6 – Considered cross-sections

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12. Consider 3 materials – **Steel S235J2, Titanium Grade 5 (Ti-6Al-4V)** and **Aluminum 7075-T6 (Fig. 7)**

Properties of **Steel S235J2**

Yield strength is $\boldsymbol{\sigma_{yield.St} = 235}$ **MPa;**
Permissible stress is
$\boldsymbol{\sigma_{perm.St} = \sigma_{yield.St} / n_{min} = 117.5}$ **MPa;**
Density is $\boldsymbol{\rho_{St} = 7850}$ **kg/m³.**

Properties of **Titanium Grade 5**

Yield strength is $\boldsymbol{\sigma_{yield.Ti} = 790}$ **MPa;**
Permissible stress is
$\boldsymbol{\sigma_{perm.Ti} = \sigma_{yield.Ti} / n_{min} = 395}$ **MPa;**
Density is $\boldsymbol{\rho_{Ti} = 4430}$ **kg/m³.**

Properties of **Aluminum 7075-T6**

Yield strength is $\boldsymbol{\sigma_{yield.Al} = 500}$ **MPa;**
Permissible stress is $\boldsymbol{\sigma_{perm.Al} = \sigma_{yield.Al} / n_{min} = 250}$ **MPa;**
Density is $\boldsymbol{\rho_{Al} = 2810}$ **kg/m³.**

a b c

Fig. 7 – Considered materials
a – Steel S235J2; b - Titanium Grade 5; c - Aluminum 7075-T6
[Images by Gnee Steel https://www.gneesteel.com/]

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13. Calculate minimum section moduli

$W_{min.St}$ = $|M_{max}|/\ \sigma_{perm.St}$ = 204.26 cm^3;
$W_{min.Ti}$ = $|M_{max}|/\ \sigma_{perm.Ti}$ = 60.76 cm^3;
$W_{min.Al}$ = $|M_{max}|/\ \sigma_{perm.Al}$ = 96 cm^3.

Diameters and areas for round beam

$d_{min.St} = [32 \cdot W_{min.St} / \pi]^{1/3}$ = 127.6 mm;
$d_{min.Ti} = [32 \cdot W_{min.Ti} / \pi]^{1/3}$ = 85.2 mm;
$d_{min.Al} = [32 \cdot W_{min.Al} / \pi]^{1/3}$ = 99.3 mm;

$A_{min.St.round} = [\pi \cdot (d_{min.St})^2] / 4$ = 128 mm^2;
$A_{min.Ti.round} = [\pi \cdot (d_{min.Ti})^2] / 4$ = 57 mm^2;
$A_{min.Al.round} = [\pi \cdot (d_{min.Al})^2] / 4$ = 77.4 mm^2.

Height and area for rectangular beam

$h_{min.St} = [32 \cdot W_{min.St} / \pi]^{1/3}$ = 134.8 mm;
$h_{min.Ti} = [32 \cdot W_{min.Ti} / \pi]^{1/3}$ = 90 mm;
$h_{min.Al} = [32 \cdot W_{min.Al} / \pi]^{1/3}$ = 104.8 mm;

$A_{min.St.rect} = h_{min.St} \cdot h_{min.St} / 2$ = 90.9 mm^2;
$A_{min.Ti.rect} = h_{min.Ti} \cdot h_{min.Ti} / 2$ = 40.5 mm^2;
$A_{min.Al.rect} = h_{min.Al} \cdot h_{min.Al} / 2$ = 54.9 mm^2.

Section modulus and area for steel I-beam (DIN 1025 / EN 10034) and assuming the other materials can also be used for I-beams

I-beam 220; W_{St} = 252 cm^3; $A_{St.I\_beam}$ = 33.4 cm^2;
I-beam 140; W_{Ti} = 77.3 cm^3; $A_{Ti.I\_beam}$ = 16.4 cm^2;
I-beam 160; W_{Al} = 109 cm^3; $A_{Al.I\_beam}$ = 20.1 cm^2.

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14. Calculate minimum volume for the selected materials and cross-sections

For round beam

$V_{min.St.round} = A_{min.St.round} \cdot (a + b + c) =$
$= 0.1024\ m^3$;

$V_{min.Ti.round} = A_{min.Ti.round} \cdot (a + b + c) =$
$= 0.0456\ m^3$;

$V_{min.Al.round} = A_{min.Al.round} \cdot (a + b + c) =$
$= 0.0619\ m^3$.

For rectangular beam

$V_{min.St.rect} = A_{min.St.rect} \cdot (a + b + c) = 0.0727\ m^3$;
$V_{min.Ti.rect} = A_{min.Ti.rect} \cdot (a + b + c) = 0.0324\ m^3$;
$V_{min.Al.rect} = A_{min.Al.rect} \cdot (a + b + c) = 0.0440\ m^3$.

For I-beam

$V_{min.St.I\_beam} = A_{St.I\_beam} \cdot (a + b + c) = 0.0267\ m^3$;
$V_{min.Ti.I\_beam} = A_{Ti.I\_beam} \cdot (a + b + c) = 0.0131\ m^3$;
$V_{min.Al.I\_beam} = A_{Al.I\_beam} \cdot (a + b + c) = 0.0161\ m^3$.

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15. Calculate minimum mass of shafts

For round beam

$m_{min.St.round} = \rho_{St} \cdot V_{min.St.round} = 804$ kg;
$m_{min.Ti.round} = \rho_{Ti} \cdot V_{min.Ti.round} = 202$ kg;
$m_{min.Al.round} = \rho_{Al} \cdot V_{min.Al.round} = 174$ kg.

For rectangular beam

$m_{min.St.rect} = \rho_{St} \cdot V_{min.St.rect} = 571$ kg;
$m_{min.Ti.rect} = \rho_{Ti} \cdot V_{min.Ti.rect} = 144$ kg;
$m_{min.Al.rect} = \rho_{Al} \cdot V_{min.Al.rect} = 124$ kg.

For I-beam

$m_{min.St.I\_beam} = \rho_{St} \cdot V_{min.St.I\_beam} = 210$ kg;
$m_{min.Ti.I\_beam} = \rho_{St} \cdot V_{min.St.I\_beam} = 58$ kg;
$m_{min.Al.I\_beam} = \rho_{St} \cdot V_{min.St.I\_beam} = 45$ kg.

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16. Material cost calculation

Prices per kg of materials are

p_{St} = 0.728 €/kg;

p_{Ti} = 5.97 €/kg;

p_{Al} = 2.955 €/kg.

Material cost for round profile is

$MC_{St.round} = m_{min.St.round} \cdot p_{St}$ = 585 €;

$MC_{Ti.round} = m_{min.Ti.round} \cdot p_{Ti}$ = 1207 €;

$MC_{Al.round} = m_{min.Al.round} \cdot p_{Al}$ = 514 €.

Material cost for rectangular beam is

$MC_{St.rect} = m_{min.St.rect} \cdot p_{St}$ = 416 €;

$MC_{Ti.rect} = m_{min.Ti.rect} \cdot p_{Ti}$ = 857 €;

$MC_{Al.rect} = m_{min.Al.rect} \cdot p_{Al}$ = 365 €.

Material cost for I-beam is

$MC_{St.I\_beam} = m_{min.St.I\_beam} \cdot p_{St}$ = 153 €;

$MC_{Ti.I\_beam} = m_{min.Ti.I\_beam} \cdot p_{Ti}$ = 347 €;

$MC_{Al.I\_beam} = m_{min.Al.I\_beam} \cdot p_{Al}$ = 134 €.

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Conclusion.

A beam made of **aluminum** is the **lightest** and the **cheapest**, out of all three materials. **The cheapest and the lightest case** is an **aluminum I-beam. Mass of aluminum I-beam** is only **22% of steel I-beam**, and **titanium I-beam** is **28% of steel I-beam**.

If comparing the three cross-sections – **the I-beam is the lightest for all three materials**. Aluminum **I-beam** is **2.7 times lighter** than a **rectangle** and **3.9 times lighter** than a **circle**.

The cost of **aluminum** beam **is 87%** of the **steel** beam cost,
and **titanium** beam is **227%** of steel beam cost.

Therefore, from considerations of just weight and price, **aluminum is the rational material**.

Thank you for your attention!

Matthias Kröger, Prof. Dr.-Ing. at TUBAF

Serhii Onyshchenko, PhD, assoc. prof. at DUT

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